Function of a Matrix

For any matrix A in $M_n(\mathbb{C})$, write $\Lambda(A)$ for the set of eigenvalues of A. Recall that, for each $A \in M_n(\mathbb{C})$, there is a unique continuous $\mathbb{C}[X]$-algebra morphism

$$\mathcal{O}(\Lambda(A)) \to M_n(\mathbb{C}),$$

where $\mathcal{O}(\Lambda(A))$ is the algebra of those functions which are holomorphic on (some open neighborhood of) $\Lambda(A)$. Recall also that this morphism is usually denoted by $f \mapsto f(A)$. (Here and in the sequel, X is an indeterminate.)

Let U be an open subset of $\mathbb{C}$, and let U' be the subset of $M_n(\mathbb{C})$ defined by the condition

$$\Lambda(A) \subset U.$$

By Rouché's Theorem, U' is open. Let f be holomorphic on U. We want to prove

THEOREM. The map $A \mapsto f(A)$ from U' to $M_n(\mathbb{C})$ is holomorphic.

Thank you to Mihai Paun for his help.

We assume (as we may) that U is a union of finitely many disjoint open disks. Let $\mathcal{O}(U)$ denote the topological algebra of holomorphic functions on U.

LEMMA 1. The map

$$F : \mathcal{O}(U) \times U' \to M_n(\mathbb{C}), \quad (f, A) \mapsto f(A)$$

is continuous.

We show that Lemma 1 implies the Theorem.

By Runge's Theorem, there is a sequence P_ν of polynomials converging to f in $\mathcal{O}(U)$. Then $F(P_\nu, ?)$ converges to $F(f, ?)$ in $C^0(U', M_n(\mathbb{C}))$. As $F(P_\nu, ?)$ is holomorphic, so is $F(f, ?)$.

The sequel is dedicated to the proof of Lemma 1.

We start with a **preliminary**.

Consider the action of the symmetric group S_n on $\mathbb{C}^n$, let $\pi : \mathbb{C}^n \to S_n \backslash \mathbb{C}^n$ be the canonical projection, and let s be the polynomial endomorphism of $\mathbb{C}^n$ whose j th component s_j is defined by the conditions $s_j(z) = c_j$ and

$$\Pi = \prod_{i=1}^{n} (X - z_i) = X^n + \sum_{j=1}^{n} c_j \, X^{n-j}. \qquad (*)$$

Then s induces a continuous map $\overline{s}$ from $S_n\backslash\mathbb{C}^n$ to $\mathbb{C}^n$.

Let r be the unique map from $\mathbb{C}^n$ to $S_n\backslash\mathbb{C}^n$ satisfying $r(s(z)) = \pi(z)$ for all z in $\mathbb{C}^n$. Then r is an inverse to $\overline{s}$, and it is continuous by Rouché's Theorem. We record this as

LEMMA 2. The above maps $\overline{s} : S_n\backslash\mathbb{C}^n \to \mathbb{C}^n$ and $r : \mathbb{C}^n \to S_n\backslash\mathbb{C}^n$ are inverse homeomorphisms.

We introduce **Hermite's Interpolation Polynomial**.

Fix a point z in the open subset

$$U^n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1, \dots, z_n \in U\}$$

of $\mathbb{C}^n$, let $w_1, \dots, w_k \in U$ be the distinct values of $z_1, \dots, z_n$, let $m_i \geq 1$ be the number of subscripts j satisfying $z_j = w_i$, let Π be as in $(*)$, and define $H(f, z, X)$ for f in $\mathcal{O}(U)$ by

$$H(f, z, X) := \sum_i T_i \left(f(X) \, \frac{(X - w_i)^{m_i}}{\Pi(X)} \right) \frac{\Pi(X)}{(X - w_i)^{m_i}} \quad ,$$

where $T_i(g(X))$ stands for the degree less than m_i Taylor polynomial of $g(X)$ at $X = w_i$.

Note that $H(f, z, X)$ is the unique degree less that n solution to the congruences

$$H(f, z, X) \equiv f(X) \bmod (X - w_i)^{m_i}, \quad 1 \leq i \leq k,$$

each congruence taking place in $\mathbb{C}[[X - w_i]]$.

In particular $H(f, z, X)$ is invariant under permutation of the z_i. So, by Lemma 2, there is a unique polynomial $H_0(f, c, X)$ such that we have $H_0(f, c, X) = H(f, z, X)$ in Notation $(*)$.

This implies:

LEMMA 3. (a) If f is a polynomial, then $H_0(f, c, X)$ is the remainder of the long division of f by Π.

(b) If Π is the characteristic polynomial of $A \in M_n(\mathbb{C})$, then we have $f(A) = H_0(f, c, A)$ by Cayley-Hamilton.

NOTE. The coefficients of $H_0(f, c, X)$ depend continuously on f, and, in view of what has been done so far, it suffices to prove that these coefficients are continuous functions of $(f, c) \in \mathcal{O}(U) \times \mathbb{C}^n$.

We introduce **Newton's Interpolation Polynomial**.

Let f be in $\mathcal{O}(U)$. For $k = 1, \dots, n$, define $f_k \in \mathcal{O}(U^k)$ inductively by

$$f_1(z_1) := f(z_1),$$

$$f_k(z_1, \dots, z_k) := \int_0^1 \frac{\partial f_{k-1}}{\partial z_{k-1}}\Big(z_1, \dots, z_{k-2}, z_{k-1} + t\,(z_k - z_{k-1})\Big)\,dt.$$

Note that, for $z_k \neq z_{k-1}$, we have

$$f_k(z_1, \dots, z_k) = \frac{f_k(z_1, \dots, z_{k-2}, z_k) - f_k(z_1, \dots, z_{k-2}, z_{k-1})}{z_k - z_{k-1}} \quad .$$

Define $N(f, z, X)$ by

$$N(f, z, X) :=$$
$$f_1(z_1) + f_2(z_1, z_2)\,(X - z_1) + \cdots + f_n(z_1, \dots, z_n)\,(X - z_1) \cdots (X - z_{n-1}).$$

We will see that $N(f, z, X)$ is equal to $H(f, z, X)$.

Let $\Pi \in \mathbb{C}[X]$ and $z, c \in \mathbb{C}^n$ be as in $(*)$.

LEMMA 4.

(a) $N(f, z, z_i) = f(z_i)$ for all i.

(b) The coefficients of $N(f, z, X)$ are invariant under permutation of the z_i.

(c) These coefficients depend continuously on $(f, z) \in \mathcal{O}(U) \times \mathbb{C}^n$.

(d) There is a unique polynomial $N_0(f, c, X)$ such that $N_0(f, c, X) = N(f, z, X)$.

(e) The coefficients of $N_0(f, c, X)$ depend continuously on $(f, c) \in \mathcal{O}(U) \times \mathbb{C}^n$.

(f) If f is a polynomial, then $N_0(f, c, X)$ is the remainder of the long division of f by Π.

Proof. We have

$$N(f, z, X) = f(z_1) + (X - z_1)\,N\big(f_2(z_1, ?), (z_2, \dots, z_n), X\big),$$

and an obvious induction completes the proof of (a). Now (b) follows from (a), and (c) is clear. Part (d) follows from Lemma 2, and (e) results from Part (c) coupled with Lemma 2 and. To prove (f), assume that f is a polynomial, say

$$f = \sum_{k=0}^{p} a_k\, X^k.$$

Then $N_0(f, c, X)$ and the remainder of the long division of f by Π are in

$$\mathbb{Z}[a_0, \dots, a_p; c_1, \dots, c_n; X].$$

So we can assume that Π has no repeated roots, and the result follows from (a). QED

In the Note after Lemma 3 we observed that it suffices to check that the coefficients of $H_0(f, c, X)$ are continuous functions of $(f, c) \in \mathcal{O}(U) \times \mathbb{C}^n$. We also remarked that these coefficients depend continuously on f.

As Lemma 4(e) says that the coefficients of $N_0(f, c, X)$ depend continuously on (f, c), it suffices to verify the equality $H_0(f, c, X) = N_0(f, c, X)$.

Lemmas 3(a) and 4(f) show this equality holds if f is a polynomial. As both sides are continuous in f, and as polynomials are dense in $\mathcal{O}(U)$ by Runge's Theorem, we are done.